

# On the Roman Bondage Number of Graphs on surfaces

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## Abstract

A Roman dominating function on a graph  $G$  is a labeling  $f : V(G) \rightarrow \{0, 1, 2\}$  such that every vertex with label 0 has a neighbor with label 2. The Roman domination number,  $\gamma_R(G)$ , of  $G$  is the minimum of  $\sum_{v \in V(G)} f(v)$  over such functions. The Roman bondage number  $b_R(G)$  is the cardinality of a smallest set of edges whose removal from  $G$  results in a graph with Roman domination number not equal to  $\gamma_R(G)$ . In this paper we obtain upper bounds on  $b_R(G)$  in terms of (a) the average degree and maximum degree, and (b) Euler characteristic, girth and maximum degree. We also show that the Roman bondage number of every graph which admits a 2-cell embedding on a surface with non negative Euler characteristic does not exceed 15.

**Keywords:** Roman domination, Roman bondage number, girth, average degree, Euler characteristic.  
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## 1 Introduction

All graphs considered in this article are finite, undirected, without loops or multiple edges. We denote the vertex set and the edge set of a graph  $G$  by  $V(G)$  and  $E(G)$ , respectively. Let  $P_n$  denote the path with  $n$  vertices. For any vertex  $x$  of a graph  $G$ ,  $N_G(x)$  denotes the set of all neighbors of  $x$  in  $G$ ,  $N_G[x] = N_G(x) \cup \{x\}$  and the degree of  $x$  is  $d_G(x) = |N_G(x)|$ . The minimum and maximum degrees of the graph  $G$  are denoted by  $\delta(G)$  and  $\Delta(G)$ , respectively. For a graph  $G$ , let  $x \in X \subseteq V(G)$ . A vertex  $y \in V(G)$  is a  $X$ -private neighbor of  $x$  if  $N_G[y] \cap X = \{x\}$ . The  $X$ -private neighborhood of  $x$ , denoted  $pn_G[x, X]$ , is the set of all  $X$ -private neighbors of  $x$ . An orientable compact 2-manifold  $\mathbb{S}_h$  or orientable surface  $\mathbb{S}_h$  (see [13]) of genus  $h$  is obtained from the sphere by adding  $h$  handles. Correspondingly, a non-orientable compact 2-manifold  $\mathbb{N}_q$  or non-orientable surface  $\mathbb{N}_q$  of genus  $q$  is obtained from the sphere by adding  $q$  crosscaps. Compact 2-manifolds are called simply surfaces throughout the paper. The Euler characteristic is defined by  $\chi(\mathbb{S}_h) = 2 - 2h$ ,  $h \geq 0$ , and  $\chi(\mathbb{N}_q) = 2 - q$ ,  $q \geq 1$ . The Euclidean plane  $\mathbb{S}_0$ , the projective plane  $\mathbb{N}_1$ , the torus  $\mathbb{S}_1$ , and the Klein bottle  $\mathbb{N}_2$  are all the surfaces of nonnegative Euler characteristic.

A dominating set for a graph  $G$  is a subset  $D \subseteq V(G)$  of vertices such that every vertex not in  $D$  is adjacent to at least one vertex in  $D$ . The minimum cardinality of a dominating set is called the domination number of  $G$  and is denoted by  $\gamma(G)$ . A variation of domination called Roman domination was introduced by ReVelle [10, 11]. Also see ReVelle and Rosing [12] for an integer programming formulation of the problem. The concept of Roman domination can be formulated in terms of graphs. A Roman dominating function

(RDF) on a graph  $G$  is a vertex labeling  $f : V(G) \rightarrow \{0, 1, 2\}$  such that every vertex with label 0 has a neighbor with label 2. For a RDF  $f$ , let  $V_i^f = \{v \in V(G) : f(v) = i\}$  for  $i = 0, 1, 2$ . Since this partition determines  $f$ , we can equivalently write  $f = (V_0^f; V_1^f; V_2^f)$ . The weight  $f(V(G))$  of a RDF  $f$  on  $G$  is the value  $\sum_{v \in V(G)} f(v)$ , which equals  $|V_1^f| + 2|V_2^f|$ . The Roman domination number of a graph  $G$ , denoted by  $\gamma_R(G)$ , is the minimum weight of a Roman dominating function on  $G$ . A function  $f = (V_0^f; V_1^f; V_2^f)$  is called a  $\gamma_R$ -function on  $G$ , if it is a Roman dominating function and  $f(V(G)) = \gamma_R(G)$ . One measure of the stability of the Roman domination number of a graph  $G$  under edge removal is the Roman bondage number  $b_R(G)$ , defined by Jafari Rad and Volkmann in [7], as the cardinality of a smallest set of edges whose removal from  $G$  results in a graph with Roman domination number not equal to  $\gamma_R(G)$ . For more information we refer the reader to [1, 2, 4, 7, 9, 15].

In this paper we obtain upper bounds for  $b_R(G)$  in terms of (a) average degree and maximum degree, and (b) Euler characteristic, girth and maximum degree. We also prove that the Roman bondage number of every graph which admits a 2-cell embedding on a surface with non negative Euler characteristic does not exceed 15.

## 2 Some known results

The following results are important for our investigations.

**Theorem A.** *Let  $G$  be a connected graph embeddable on a surface  $\mathbb{M}$  whose Euler characteristic  $\chi(\mathbb{M})$  is nonnegative and let  $\delta(G) \geq 5$ . Then  $G$  contains an edge  $e = xy$  with  $d_G(x) + d_G(y) \leq 11$  if one of the following holds:*

- (i) (Wernicke [18] and Sanders [17], respectively)  $\mathbb{M} \in \{\mathbb{S}_0, \mathbb{N}_1\}$ .
- (ii) (Jendrol' and Voss [6])  $\mathbb{M} \in \{\mathbb{S}_1, \mathbb{N}_2\}$  and  $\Delta(G) \geq 7$ .

**Lemma B** (Rad and Volkmann [8]). *If  $G$  is a graph, then  $\gamma_R(G) \leq \gamma_R(G - e) \leq \gamma_R(G) + 1$  for any edge  $e \in E(G)$ .*

According to the effects of vertex removal on the Roman domination number of a graph  $G$ , let

- $V_R^+(G) = \{v \in V(G) \mid \gamma_R(G - v) > \gamma_R(G)\}$ ,
- $V_R^-(G) = \{v \in V(G) \mid \gamma_R(G - v) < \gamma_R(G)\}$ ,
- $V_R^0(G) = \{v \in V(G) \mid \gamma_R(G - v) = \gamma_R(G)\}$ .

Clearly  $\{V_R^-(G), V_R^0(G), V_R^+(G)\}$  is a partition of  $V(G)$ .

**Theorem C** (Rad and Volkmann [8]). *Let  $G$  be a graph of order at least 2.*

- (i) *If  $v \in V_R^+(G)$  then for every  $\gamma_R$ -function  $f = (V_0^f; V_1^f; V_2^f)$  on  $G$ ,  $|pn_G[v, V_2^f] \cap V_0^f| \geq 3$  and  $f(v) = 2$ .*
- (ii) *For any vertex  $u \in V(G)$ ,  $\gamma_R(G) - 1 \leq \gamma_R(G - u)$ .*

**Theorem D** (Hansberg, Rad and Volkmann [5]). *Let  $v$  be a vertex of a graph  $G$ . Then  $\gamma_R(G - v) < \gamma_R(G)$  if and only if there is a  $\gamma_R$ -function  $f = (V_0, V_1, V_2)$  on  $G$  such that  $v \in V_1$ .*

**Theorem E** (Rad and Volkmann [7]). *If  $G$  is a graph, and  $x, y, z$  is a path of length 2 in  $G$ , then*

$$b_R(G) \leq d_G(x) + d_G(y) + d_G(z) - 3 - |N_G(x) \cap N_G(y)|.$$

The average degree  $ad(G)$  of a graph  $G$  is defined as  $ad(G) = 2|E(G)|/|V(G)|$ .

**Theorem F** (Hartnell and Rall [3]). *For any connected nontrivial graph  $G$ , there exists a pair of vertices, say  $u$  and  $v$ , that are either adjacent or at distance 2 from each other, with the property that  $d_G(u) + d_G(v) \leq 2ad(G)$ .*

The girth of a graph  $G$  is the length of a shortest cycle in  $G$ ; the girth of a forest is  $\infty$ .

**Lemma G** (Samodivkin [16]). *Let  $G$  be a connected graph embeddable on a surface  $\mathbb{M}$  whose Euler characteristic  $\chi$  is as large as possible and let the girth of  $G$  is  $k < \infty$ . Then:*

$$ad(G) \leq \frac{2k}{k-2} \left(1 - \frac{\chi}{|V(G)|}\right).$$

Given a graph  $G$  of order  $n$ , let  $\widehat{G}$  be the graph of order  $5n$  obtained from  $G$  by attaching the central vertex of a copy of  $P_5$ , to each vertex of  $G$ .

**Lemma H** (S.Akbari, M. Khatirinejad and S. Qajar [1]). *Let  $G$  be a graph of order  $n$ ,  $n \geq 2$ . Then  $\gamma(\widehat{G}) = 2n$ ,  $\gamma_R(\widehat{G}) = 4n$  and  $b_R(\widehat{G}) = \delta(G) + 2$ .*

### 3 Upper bounds

A graph  $G$  of order at least two is Roman domination vertex critical if removing any vertex of  $G$  decreases the Roman domination number. By  $\mathcal{RCV}$  we denote the class of all Roman domination vertex critical graphs. Results on this class can be found in Rad and Volkmann [8] and Hansberg et al. [5].

**Theorem 1.** *Let  $G$  be a connected graph.*

- (i) *If  $V_R^-(G) \neq V(G)$  then  $b_R(G) \leq \min\{d_G(u) - \gamma_R(G - u) + \gamma_R(G) \mid u \in V_R^0(G) \cup V_R^+(G)\} \leq \Delta(G)$ .*
- (ii) *If  $b_R(G) > \Delta(G)$  then  $G$  is in  $\mathcal{RCV}$ .*

*Remark 2.* Let  $G$  be any connected graph of order  $n \geq 2$ . Denote by  $S$  the set of all vertices of  $\widehat{G}$  each of which is adjacent to a vertex of degree 1. Then  $f = (V(\widehat{G}) - S; \emptyset; S)$  is a RDF on  $\widehat{G}$ . Since the weight of  $f$  is  $4n$ , by Lemma H it follows that  $f$  is a  $\gamma_R$ -function on  $\widehat{G}$ . Theorem C(i) now implies  $V(\widehat{G}) = V_R^-(\widehat{G}) \cup V_R^0(\widehat{G})$ . Since  $\gamma_R(P_5) = 4$  and since the central vertex of  $P_5$  is in  $V_R^0(P_5)$ ,  $V(G) \subset V_R^0(\widehat{G})$ . Labelling the vertices of each  $P_5$  of  $\widehat{G}$  with  $(1, 1, 0, 2, 0)$  yields a  $\gamma_R$ -function on  $\widehat{G}$ . It follows by Theorem D that  $V(G) = V_R^0(\widehat{G})$ . All this together with Lemma H shows that the bound in Theorem 1(i) is attainable for all graphs  $\widehat{G}$ . Furthermore, for any graph  $\widehat{G}$  the bound in Theorem E is attainable too.

To prove Theorem 1, we need the following lemma:

**Lemma 3.** *Let  $G$  be a connected graph. For any subset  $U \subsetneq V(G)$ , let  $E_U$  denote the set of all edges each of which joins  $U$  and  $V(G) - U$ .*

- (i) *If  $v \in V_R^0(G) \cup V_R^+(G)$  then  $\gamma_R(G - E_{\{v\}}) > \gamma_R(G)$ .*
- (ii) *If  $x \in V_R^+(G)$  then  $1 \leq \gamma_R(G - x) - \gamma_R(G) \leq d_G(x) - 2$  and for any subset  $S \subseteq E_{\{x\}}$  with  $|S| \geq d_G(x) - \gamma_R(G - v) + \gamma_R(G)$ ,  $\gamma_R(G - S) > \gamma_R(G)$ .*

*Proof.* (i) We have  $\gamma_R(G - E_{\{v\}}) = \gamma_R(G - v) + 1 > \gamma_R(G)$ .

(ii) Denote  $p = \gamma_R(G - x) - \gamma_R(G)$ . Let  $f$  be any  $\gamma_R$ -function on  $G$ . Since  $p > 0$ , by Theorem C(i) it follows that  $f(x) = 2$ . Hence  $h = (V_0^f - N_G(x); V_1^f \cup (N_G(x) - V_2^f); V_2^f - \{x\})$  is a RDF on  $G - x$ . But then  $\gamma_R(G) + p = \gamma_R(G - x) \leq h(V(G - x)) \leq \gamma_R(G) + d_G(x) - 2$ . Hence  $1 \leq p \leq d_G(x) - 2$ . For any set  $S \subseteq E_{\{x\}}$  with  $|S| \geq d_G(x) - p$  we have  $\gamma_R(G - S) \geq \gamma_R(G - E_{\{x\}}) - |E_{\{x\}}| + |S| \geq (\gamma_R(G - x) + 1) - d_G(x) + (d_G(x) - p) = \gamma_R(G) + 1$ , where the first inequality follows from Lemma B.  $\square$

*Proof of Theorem 1.* (i) The result follows immediately by Lemma 3.

(ii) Immediately by (i).  $\square$

Rad and Volkmann [9] as well as Akbari et al. [1] gave upper bounds on the Roman bondage number of planar graphs. Upper bounds on the Roman bondage number of graphs 2-cell embeddable on topological surfaces in terms of orientable/non orientable genus and maximum degree, are obtained by the present author in [15].

**Theorem 4.** *Let  $G$  be a connected graph with  $\Delta(G) \geq 2$ .*

(i) *Then  $b_R(G) \leq 2ad(G) + \Delta(G) - 3$ .*

(ii) *Let  $G$  be embeddable on a surface  $\mathbb{M}$  whose Euler characteristic  $\chi$  is as large as possible. If  $G$  has order  $n$  and girth  $k < \infty$  then:*

$$b_R(G) \leq \frac{4k}{k-2} \left(1 - \frac{\chi}{n}\right) + \Delta(G) - 3 \leq -\frac{12\chi}{n} + \Delta(G) + 9.$$

*Proof.* (i) If  $G$  is a complete graph then the result is obvious. Hence we may assume  $G$  has nonadjacent vertices. Theorem F implies that there are 2 vertices, say  $x$  and  $y$ , that are either adjacent or at distance 2 from each other, with the property that  $d_G(x) + d_G(y) \leq 2ad(G)$ . Since  $G$  is connected and  $\Delta(G) \geq 2$ , there is a vertex  $z$  such that  $xyz$  or  $xzy$  is a path. In either case by Theorem E we have  $b_R(G) \leq d_G(x) + d_G(y) + d_G(z) - 3 \leq 2ad(G) + \Delta(G) - 3$ .

(ii) Lemma G and (i) together imply the result.  $\square$

*Remark 5.* Let  $\mathbb{M}$  be a surface. Denote  $\delta_{max}^{\mathbb{M}} = \max\{\delta(H) \mid \text{a graph } H \text{ is 2-cell embedded in } \mathbb{M}\}$ . Let  $G$  be a connected graph 2-cell embeddable on  $\mathbb{M}$  and  $\delta(G) = \delta_{max}^{\mathbb{M}}$ . By Lemma H it immediately follows  $b_R(\hat{G}) = \delta_{max}^{\mathbb{M}} + 2$ . Note that (a) if  $\chi(\mathbb{M}) \leq 1$  then  $\delta_{max}^{\mathbb{M}} \leq \left\lfloor (5 + \sqrt{49 - 24\chi(\mathbb{M})})/2 \right\rfloor$  (see Sachs [14], pp. 226-227), and (b) it is well known that  $\delta_{max}^{S_0} = \delta_{max}^{N_1} = 5$  and  $\delta_{max}^{N_2} = \delta_{max}^{N_3} = \delta_{max}^{S_1} = 6$ .

In [1], Akbari, Khatirinejad and Qajar recently prove that  $b_R(G) \leq 15$  provided  $G$  is a planar graph. As the next result shows, more is true.

**Theorem 6.** *Let  $G$  be a connected graph 2-cell embedded on a surface  $\mathbb{M}$  with non negative Euler characteristic and let  $\Delta(G) \geq 2$ . Then  $b_R(G) \leq 15$ .*

*Proof.* If  $2 \leq \Delta(G) \leq 6$  then  $b_R(G) \leq 3\Delta(G) - 3 \leq 15$ , because of Theorem E. So, assume  $\Delta(G) \geq 7$ . Denote  $V_{\leq 5} = \{v \in V(G) \mid d_G(v) \leq 5\}$  and  $G_{\geq 6} = G - V_{\leq 5}$ . Since  $\chi(\mathbb{M}) \geq 0$ ,  $\delta(G) \leq 6$  (see Remark 5). If  $\delta(G) = 6$  then  $G$  is a 6-regular triangulation on the torus or in the Klein Bottle, a contradiction with  $\Delta(G) \geq 7$ . So,  $\delta(G) \leq 5$  and then  $V_{\leq 5}$  is not empty. Since  $G_{\geq 6}$  is embedded without crossings on  $\mathbb{M}$  and  $\chi(\mathbb{M}) \geq 0$ , there is a vertex  $u \in V(G_{\geq 6})$  with  $d_{G_{\geq 6}}(u) \leq 6$ . If  $u$  has exactly 2 neighbors belonging to  $V_{\leq 5}$  then again by Theorem E,  $b_R(G) \leq 15$ . Now let all  $v_1, v_2, v_3 \in V_{\leq 5}$  be adjacent to  $u$ . Denote by  $E_1$  the set of all edges of  $G$  which are incident to at least one of  $v_1, v_2$  and  $v_3$ . Since  $v_1, v_2$  and  $v_3$  are isolated vertices in  $G - E_1$ , for any  $\gamma_{PR}$ -function  $g$  on  $G - E_1$ ,  $g(v_1) = g(v_2) = g(v_3) = 1$ . Define now  $f : V(G) \rightarrow \{0, 1, 2\}$  by  $f(v_1) = f(v_2) = f(v_3) = 0$ ,  $f(u) = 2$  and  $f(v) = g(v)$  for every  $v \in V(G) - \{u, v_1, v_2, v_3\}$ . Clearly  $f$  is a RDF on  $G$  with  $\gamma_R(G) \leq f(V(G)) < g(V(G - E_1)) = \gamma_R(G - E_1)$ . Thus,  $b_R(G) \leq |E_1| \leq d_G(v_1) + d_G(v_2) + d_G(v_3) \leq 15$ .

So, it remains to consider the case where each vertex of degree at most 6 in  $G_{\geq 6}$  has no more than one neighbor in  $V_{\leq 5}$ . It immediately follows that  $\delta(G_{\geq 6}) \geq 5$ . First assume  $\delta(G_{\geq 6}) = 5$ . By Theorem A it follows that there is an edge  $xy \in E(G_{\geq 6})$  such that  $d_{G_{\geq 6}}(x) + d_{G_{\geq 6}}(y) \leq 11$ . Hence  $d_G(x) + d_G(y) \leq 13$ . Let without loss of generality  $d_{G_{\geq 6}}(x) \leq d_{G_{\geq 6}}(y)$ . Then  $x$  has exactly one neighbor in  $V_{\leq 5}$ , say  $v$ . By Theorem E applied to the path  $v, x, y$  we have  $b_R(G) \leq 5 + 13 - 3 = 15$ . Now let  $\delta(G_{\geq 6}) \geq 6$ . But then  $G_{\geq 6}$  is a 6-regular triangulation on the torus or in the Klein bottle. Since  $\Delta(G) \geq 7$ ,  $G \neq G_{\geq 6}$  and there is a path  $x, y, z$  in  $G$ , where  $d_G(z) \leq 5$ , and both  $x$  and  $y$  are in  $V(G_{\geq 6})$ . Since clearly  $|N(x) \cap N(y)| \geq 2$ , again using Theorem E we obtain  $b_{PR}(G) \leq 7 + 7 + 5 - 3 - 2 = 14$ .  $\square$

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